Orbits Visiting Finite Sets

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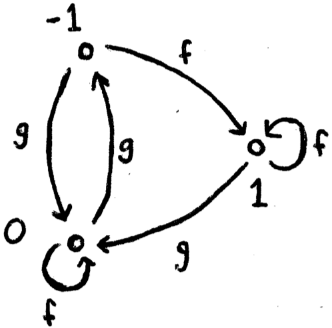
Joint work with Mike Zieve



Visiting a finite set

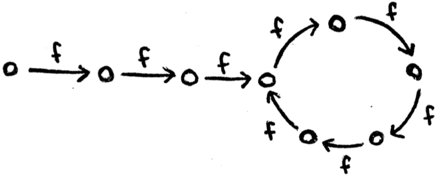
The set-up:

- K be a field
- $f(x) \in K(x)$ be a rational function
- $p \in K$ be a point
- $S \subseteq K$ a finite set.

When does the f-orbit of p visit the finite set S?

$$\{n : f^n(p) \in S\} = ?$$

When does the f-orbit of p visit S?



- Typically finitely often, probably never.
- However, if the f-orbit of p visits S more than $|S|$ times, then it does so infinitely often!

$$\{n : f^n(p) \in S\} = \text{finite union of arithmetic progressions}$$

Visiting a finite set again

Let's make this question more interesting by replacing all iterates of a single rational function

$$\langle f \rangle = \{f^n : n \geq 0\}$$

with all words in a finite set of rational functions

$$M = \langle f_1, f_2, \ldots, f_m \rangle = \{f_{i_1} f_{i_2} \cdots f_{i_k} : k \geq 0\}$$
$$M\text{-orbit of } p = M(p) = \{w(p) : w \in M\}.$$

When does the M-orbit of p visit the finite set S?

$$\{w \in M : w(p) \in S\} = ?$$

When does the M-orbit of p visit S?

Theorem (H, Zieve)

Let K be a field and let $M = \langle f_1, \ldots, f_m \rangle$ with $f_k(x) \in K(x)$ such that $\deg(f_k) \geq 2$. If $p \in K$ and $S \subseteq K$ is a finite set, then

$$\{w \in M : w(p) \in S\}$$

*is a **regular language**.*

Regular languages

A **regular expression** is a type of pattern used to describe a collection of words (= sequences of letters from an alphabet.)

{Regular expressions} is the closure of the alphabet under

- concatentation $(w_1 w_2)$
- disjunction $w_1 | w_2$
- Kleene star w^*

Ex. Say our alphabet consists of two letters f and g.

- $(f|g)^* f$ describes all words "starting" with f
- $(f^* g f^* g f^*)^*$ describes all words with an even number of g's

A **regular language** is the collection of all words described by a regular expression.

Example

Theorem (H, Zieve)

*Let K be a field and let $M = \langle f_1, \ldots, f_m \rangle$ with $f_k(x) \in K(x)$ such that $\deg(f_k) \geq 2$. If $p \in K$ and $S \subseteq K$ is a finite set, then $\{w \in M : w(p) \in S\}$ is a **regular language**.*

Ex. Let $p = 2$, $S = \{1, 4\}$, and $M = \langle f, g \rangle$ where

$$f(x) = x^2, \quad g(x) = \frac{-11x^3 + 57x^2 - 70x + 24}{24}.$$

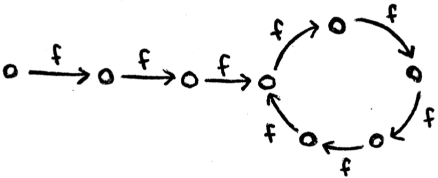
Then $\{w \in M : w(p) \in S\}$ is the regular language described by

$$(f^* g f^* g f^*)^* g \mid (fg)^* f$$

Preperiodic points

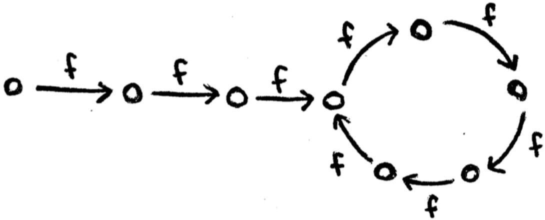
Say $p \in K$ is **preperiodic** under $f(x)$ if for some $j \geq 0, k \geq 1$

$$f^{j+k}(p) = f^j(p).$$



- What does it mean for a point p to be preperiodic under a finitely generated dynamical system $M = \langle f_1, f_2, \ldots, f_m \rangle$?

Preperiodic = Finite orbit



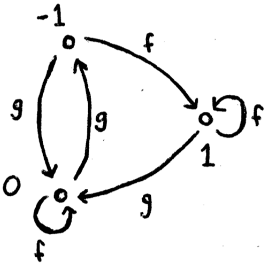
- p preperiodic under $f \iff f$-orbit of p is finite.
- Say p is **preperiodic** under $M = \langle f_1, f_2, \ldots, f_m \rangle$ if the M-orbit of p is finite.

Preperiodic = Finite orbit

Ex: Let $M = \langle f, g \rangle$ where

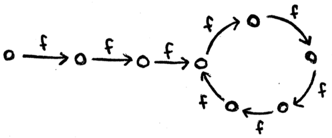
$$f(x) = x^2 \qquad g(x) = x^2 - 1.$$

Then 0 is preperiodic under M.

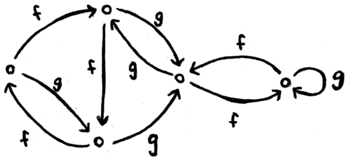


Shape of finite orbits

- Finite orbits of $\langle f \rangle$ all have the same "shape".



- But there is a wide range of finite orbit shapes for $M = \langle f_1, f_2, \ldots, f_m \rangle$.

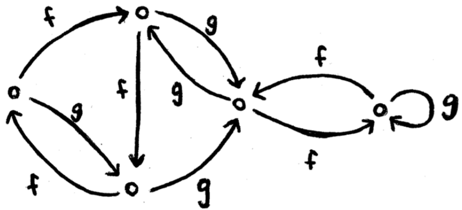


- These are **deterministic finite automata (DFA)**!

Deterministic finite automata (DFA)

DFA over an **alphabet** $f_1, f_2, \ldots, f_m$ is a finite directed graph A with a distinguished **start state** p and set S of **accept states**.

- vertices = **states**
- labelled edges = **transitions**.
- For each letter f_k there is exactly one transition labelled f_k out of each state.



Kleene's Theorem

An automata A is a simple machine for processing words.

- Beginning at the start state p read w one letter at a time and transition accordingly.
- If we end at a state in S, then A **accepts** w.
- Language of A is the set of all words $L(A)$ accepted by A.

Theorem (Kleene's Theorem)

- *If A is a DFA, then $L(A)$ is a regular language.*
- *If L is a regular language, then there is a DFA A such that $L = L(A)$.*

Kleene's theorem

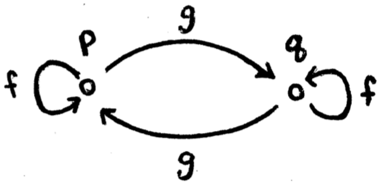
Theorem (Kleene's Theorem)

- *If A is a DFA, then $L(A)$ is a regular language.*
- *If L is a regular language, then there is a DFA A such that $L = L(A)$.*

Ex. The regular language L described by the regular expression

$$(f^*gf^*g)^*(f^*gf^*) = \text{ all words with an odd number of } g\text{'s},$$

is accepted by A shown below with $S = \{q\}$.



Proof sketch

Theorem (H, Zieve)

Let K be a field and let $M = \langle f_1, \dots, f_m \rangle$ with $f_k(x) \in K(x)$ such that $\deg(f_k) \geq 2$. If $p \in K$ and $S \subseteq K$ is a finite set, then

$$\{w \in M : w(p) \in S\}$$

is a ***regular language***.

Proof sketch

- Let h be a height function on $\mathbb{P}^1(K)$
- $(\deg(f_k) \geq 2)$ There exists a B such that
  - $h(s) \leq B$ for $s \in S$ or $s = p$,
  - $h(f_k(q)) > h(q)$ whenever $h(q) > B$.
- Let A be the finite automaton with states consisting of
  - all $q \in K$ with $h(q) \leq B$
  - a “dead state” D
- Transition labelled f_k from q to D iff $h(f_k(q)) > B$.
  - D only transitions to itself.
- $L(A) = \{w \in M : w(p) \in S\}$. □

Further Questions

- Given $M = \langle f_1, f_2, \ldots, f_m \rangle$ can we characterize the auotmata A for which there are A-periodic points?
  - Using interpolation, all automata possible for some choice of rational functions.
- If M has good reduction at a prime ℓ and p is an A-periodic point, how does the period of p modulo ℓ relate to A?
  - Does this lead to new dynamical unit constructions?
- Suppose that f is a continuous endomorphism of a real interval X. Sharkovskii proved that if f has a 3-periodic point in X, then it has periodic points of all periods in X.
  - Does this generalize to the non-cyclic setting?

Thank you!